THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH4240 - Stochastic Processes - 2023/24 Term 2

Homework 5
Due: March 22 Friday (11:59pm), 2024

All questions are selected from the textbook. Submit your answers in a single PDF file **via Blackboard online** to ONLY the Compulsory Part. Reference solutions to both parts will be provided after grading.

Compulsory Part
Exercises (Chapter 2, Page 80): 14, 15, 19, 20, 21, 22, 23

Optional Part
Exercises (Chapter 2, Page 80): 11, 12, 13, 16, 17, 18

Compulsory Part

14. Solution. Suppose that the stationary distribution π exists. Then $\pi P = P$ and $\sum_{x=0}^{\infty} \pi(x) = 1$ imply that

$$\begin{aligned}
\pi(0) &= \sum_{x=0}^{\infty} \pi(x)P(x,0) = (1-p)\sum_{x=0}^{\infty} \pi(x) = 1-p, \\
\pi(1) &= \pi(0)P(0,1) = (1-p)p, \\
\pi(2) &= \pi(1)P(1,2) = (1-p)p^2, \\
&\cdots
\end{aligned}$$

By induction, $\pi(n) = (1-p)p^n$, $n \geq 0$.

On the other hand, check that above π satisfies both $\sum_{n=0}^{\infty} \pi(n) = 1$ and $\pi(n) = \sum_{m=0}^{\infty} \pi(m)P(m,n)$, $n \geq 0$. Hence $\pi = (1-p, (1-p)p, (1-p)p^2, \cdots)$ is the unique stationary distribution.

15. Solution. Let $\mathcal{S} = \{1, 2, \ldots, d\}$ be the state space. Since all states are in a finite irreducible closed set, they are positive recurrent. Thus the stationary distribution is unique (page 68, Corollary 7).

Let $\pi(x) = \frac{1}{d}$ for all $x \in \mathcal{S}$. Then it is a probability vector since $\sum_{x=1}^{d} \pi(x) = 1$. Moreover, for all $y \in \mathcal{S}$,

$$\sum_{x=1}^{d} \pi(x)P(x,y) = \sum_{x=1}^{d} \frac{1}{d}P(x,y) = \frac{1}{d} = \pi(y).$$

This shows π is the unique stationary distribution we want.

19. Solution. **(a)** For the irreducible closed set $\{1, 2, 3\}$, its transition matrix is given by $\begin{bmatrix} 0\,1\,0 \\ 0\,0\,1 \\ 1\,0\,0 \end{bmatrix}$. This matrix is doubly stochastic. By Q15, the stationary distribution concentrated on $\{1, 2, 3\}$ is given by $(0, 1/3, 1/3, 1/3, 0, 0, 0)$.

For the irreducible closed set $\{4, 5, 6\}$, its transition matrix is given by $\begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. This matrix is doubly stochastic. By Q15, the stationary distribution concentrated on $\{4, 5, 6\}$ is given by $(0, 0, 0, 0, 1/3, 1/3, 1/3)$.

(b) We use Theorem 1 in textbook, page 58. If y is recurrent and $\pi(y)$ is the stationary distribution concentrared on the corresponding irreducible closed set,

$$\lim_{n\to\infty} \frac{G_n(x,y)}{n} = \frac{\rho_{xy}}{m_y} = \rho_{xy} \cdot \pi(y).$$

If y is transient, it is clear that $\lim_{n\to\infty}\frac{G_n(x,y)}{n}=0$. As all ρ_{xy} and $\pi(y)$ are computed before, we have

$$[\lim_{n\to\infty}\frac{G_n(x,y)}{n}]_{0\le x,y\le 6}=\begin{bmatrix} 0 & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{12} & \frac{1}{12} & \frac{1}{12} \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix}.$$

20. Solution. (a) For the irreducible closed set $\{0,1\}$, its transition matrix is given by $P_1=\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}$. Let $\pi_1=(\pi_1(0),\pi_1(1))$ and then solve

$$\begin{cases} \pi_1 P_1=\pi_1, \\ \pi_1(0)+\pi_1(1)=1. \end{cases}$$

We get $\pi_1=(\frac{2}{5},\frac{3}{5})$. Hence the stationary distribution concentrated on $\{0,1\}$ is given by $(\frac{2}{5},\frac{3}{5},0,0,0,0)$.

For the irreducible closed set $\{2,4\}$, its transition matrix is given by $P_2=\begin{bmatrix} \frac{1}{8} & \frac{7}{8} \\ \frac{3}{4} & \frac{1}{4} \end{bmatrix}$. Let $\pi_2=(\pi_2(2),\pi_2(4))$ and then solve

$$\begin{cases} \pi_2 P_2=\pi_2, \\ \pi_2(2)+\pi_2(4)=1. \end{cases}$$

We get $\pi_2=(\frac{6}{13},\frac{7}{13})$. Hence the stationary distribution concentrated on $\{2,4\}$ is given by $(0,0,\frac{6}{13},0,\frac{7}{13},0)$.

(b) We use Theorem 1 in textbook, page 58. If y is recurrent and $\pi(y)$ is the stationary distribution concentrared on the corresponding irreducible closed set,

$$\lim_{n\to\infty}\frac{G_n(x,y)}{n}=\frac{\rho_{xy}}{m_y}=\rho_{xy}\cdot\pi(y).$$

If y is transient, it is clear that $\lim_{n\to\infty}\frac{G_n(x,y)}{n}=0$. As all ρ_{xy} and $\pi(y)$ are computed before, we have

$$[\lim_{n\to\infty}\frac{G_n(x,y)}{n}]_{0\le x,y\le 5}=\begin{bmatrix} \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ \frac{2}{5} & \frac{3}{5} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{6}{13} & 0 & \frac{7}{13} & 0 \\ \frac{14}{55} & \frac{21}{55} & \frac{24}{143} & 0 & \frac{28}{143} & 0 \\ 0 & 0 & \frac{6}{13} & 0 & \frac{7}{13} & 0 \\ \frac{12}{55} & \frac{18}{55} & \frac{30}{143} & 0 & \frac{35}{143} & 0 \end{bmatrix}.$$

21. Solution. The stationary distribution is given by Q7(a):

$$\pi = (\pi(0), \pi(1), \pi(2), \pi(3), \pi(4)) = (\frac{1}{16}, \frac{1}{4}, \frac{3}{8}, \frac{1}{4}, \frac{1}{16}).$$

The period of the chain is 2.

(a) It follows from Theorem 7 in page 73 that for n large and even

$$\big(P_0(X_n = x)\big)_{0\le x\le 4} = \big(P^n(0,x)\big)_{0\le x\le 4} \approx (2\pi(0), 0, 2\pi(2), 0, 2\pi(4)) = (\frac{1}{8}, 0, \frac{3}{4}, 0, \frac{1}{8}).$$

(a) It follows from Theorem 7 in page 73 that for n large and odd

$$\big(P_0(X_n = x)\big)_{0\le x\le 4} = \big(P^n(0,x)\big)_{0\le x\le 4} \approx (0, 2\pi(1), 0, 2\pi(3), 0,) = (0, \frac{1}{2}, 0, \frac{1}{2}, 0).$$

22. Solution. **(a)** Denote $i \to j$ if $P(i,j) > 0$, where P is the transition probability. Note that in this matrix

$$0 \to 2 \to 1 \to 0,$$

the chain is irreducible.

(b) Note that

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}; \quad P^3 = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{2} \end{bmatrix},$$

thus $P^2(0,0) > 0$ and $P^3(0,0) > 0$, the period of 0 is given by $d_0 = g.c.d.\{n : P^n(0,0) > 0\} = 1$.

(c) Let π be the stationary distribution. Then $\pi(0) + \pi(1) + \pi(2) = 1$. Solve the equation $\pi P = \pi$. We have $\pi = (\frac{2}{5}, \frac{1}{5}, \frac{2}{5})$.

23. Solution. **(a)** Since

$$0 \to 1 \to 3 \to 0 \to 2 \to 4 \to 0,$$

the chain is irreducible.

(b) Since $P(0,0) = 0$, $P^2(0,0) = 0$, $P^3(0,0) \ge P(0,1)P(1,3)P(3,0) > 0$, together with $P^4 = P$, the period of the chain is 3.

(c) Let π be the stationary distribution. Then $\pi(0)+\pi(1)+\pi(2)+\pi(3)+\pi(4) = 1$. Solve the equation $\pi P = \pi$. We have $\pi = (\frac{1}{2}, \frac{1}{9}, \frac{2}{9}, \frac{1}{12}, \frac{1}{4})$.

Optional Part

11. Proof. We use induction on n. For $n = 0$, X_0 has a Poisson distribution with parameter $t = tp^0 + \frac{\lambda}{q}(1 - p^0)$. Suppose that X_n has a Poisson distribution with parameter $tp^n + \frac{\lambda}{q}(1 - p^n)$ for some $n \geq 0$. Then applying the result in page 54 of the textbook, $R(X_n)$ has a Poisson distribution with parameter $p(tp^n + \frac{\lambda}{q}(1 - p^n)) = tp^{n+1} + \frac{\lambda}{q}(1 - p^{n+1}) - \lambda$. Set $\mu_n = tp^{n+1} + \frac{\lambda}{q}(1 - p^{n+1}) - \lambda$. Then for $x \geq 0$,

$$\begin{aligned}
P(X_{n+1} = x) &= P(\xi_{n+1} + R(X_n)) \\
&= \sum_{y=0}^{x} P(R(X_n) = y, \xi_{n+1} = x - y) \\
&= \sum_{y=0}^{x} P(R(X_n) = y)P(\xi_{n+1} = x - y) \\
&= \sum_{y=0}^{x} \frac{\mu_n^y e^{-y}}{y!} \frac{\lambda^{x-y} e^{-(x-y)}}{(x-y)!} \\
&= \frac{e^{-x}}{x!} \sum_{y=0}^{x} \binom{x}{y} \mu_n^y \lambda^{x-y} \\
&= \frac{(\mu_n + \lambda)^x e^{-x}}{x!}
\end{aligned}$$

which shows that X_{n+1} has the Poisson distribution with parameter $\mu_n + \lambda = tp^{n+1} + \frac{\lambda}{q}(1 - p^{n+1})$. By induction, X_n has the indicated Poisson distribution.

12. Proof. We use induction on n. For $n = 0$, $E_x(X_0) = x = xp^0 + \frac{\lambda}{q}(1 - p^0)$. Suppose that $E_x(X_n) = xp^n + \frac{\lambda}{q}(1-p^n)$ for some $n \geq 0$. Note that ξ_{n+1} has the Poisson distribution with parameter λ. We have

$$E_x(\xi_{n+1}) = \sum_{x=0}^{\infty} x \frac{\lambda^x e^{-\lambda}}{x!} = \lambda \sum_{x=1}^{\infty} \frac{\lambda^{x-1} e^{-\lambda}}{(x-1)!} = \lambda.$$

By the Total Expectation Formula and Markov property,

$$\begin{aligned}
E_x(R(X_n)) &= \sum_{y=0}^{\infty} E(R(X_n) \mid X_n = y) P_x(X_n = y) \\
&= \sum_{y=0}^{\infty} py \cdot P_x(X_n = y) \\
&= pE_x(X_n) = xp^{n+1} + \frac{\lambda}{q}(p - p^{n+1}).
\end{aligned}$$

Hence

$$E_x(X_{n+1}) = E_x(\xi_{n+1} + R(X_n)) = E_x(\xi_{n+1}) + E_x(R(X_n)) = xp^{n+1} + \frac{\lambda}{q}(1 - p^{n+1}).$$

13. Solution. Since X_0 has the stationary distribution π, X_n has the same distribution π for any $n \geq 0$. For $m \geq 0$ and $n \geq 0$, by the Total Expectation Formula, the result of Q12 and (16),

$$\begin{aligned} E(X_m X_{m+n}) &= \sum_{x=0}^{\infty} E(X_m X_{m+n} \mid X_m = x) P(X_m = x) \\ &= \sum_{x=0}^{\infty} x E(X_{m+n} \mid X_m = x) P(X_m = x) = \sum_{x=0}^{\infty} x E_x(X_n) \pi(x) \\ &= \sum_{x=1}^{\infty} \frac{(\lambda/q)^x e^{-\lambda/q}}{(x-1)!} \left(x p^n + \frac{\lambda}{q}(1 - p^n) \right) \\ &= \frac{\lambda}{q} \sum_{x=1}^{\infty} \frac{(\lambda/q)^{x-1} e^{-\lambda/q}}{(x-1)!} \left((x-1)p^n + p^n + \frac{\lambda}{q}(1 - p^n) \right) \\ &= \frac{\lambda}{q} \left(p^n + \frac{\lambda}{q}(1 - p^n) \right) + \left(\frac{\lambda}{q} \right)^2 p^n \\ &= \frac{\lambda}{q} \left(p^n + \frac{\lambda}{q} \right). \end{aligned}$$

Hence

$$\begin{aligned} \operatorname{cov}(X_m, X_{m+n}) &= E(X_m X_{m+n}) - E(X_m) E(X_{m+n}) \\ &= \frac{\lambda}{q} \left(p^n + \frac{\lambda}{q} \right) - (E(X_0))^2 \\ &= \frac{\lambda}{q} \left(p^n + \frac{\lambda}{q} \right) - \left(\sum_{x=0}^{\infty} x \frac{(\lambda/q)^x e^{-\lambda/q}}{x!} \right)^2 \\ &= \frac{\lambda}{q} \left(p^n + \frac{\lambda}{q} \right) - \left(\frac{\lambda}{q} \right)^2 = \frac{\lambda p^n}{q}. \end{aligned}$$

16. Proof. For any $x \in \mathcal{S}$,

$$\sum_{y \in \mathcal{S}} Q(x, y) = 1 - p_x + \sum_{y \in \mathcal{S}: y \neq x} p_x P(x, y) = 1 - p_x + p_x \sum_{y \in \mathcal{S}: y \neq x} P(x, y) = 1 - p_x + p_x = 1.$$

Hence Q is the transition function of a Markov chain.

For $x, y \in \mathcal{S}$, since x leads to y in the Markov chain with respect to the transition function P, there exists a positive integer n, and $x_1, x_2, \cdots x_{n-1} \in \mathcal{S}$ such that

$$P(x, x_1) P(x_1, x_2) \cdots P(x_{n-1}, y) > 0.$$

This implies

$$Q(x, x_1) Q(x_1, x_2) \cdots Q(x_{n-1}, y) = p_x p_{x_1} \cdots p_{x_{n-1}} P(x, x_1) P(x_1, x_2) \cdots P(x_{n-1}, y) > 0.$$

Thus x leads to y in the Markov chain with respect to the transition function Q. Therefore the new chain is irreducible.

Since the state space $\mathcal{S}$ is finite, all states are positive recurrent, hence the new chain has a unique stationary distribution (page 68, Corollary 7).

Let $\pi'(x) = \frac{p_x^{-1}\pi(x)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)}$, $x \in \mathcal{S}$. Then clearly $\pi'(x) \geq 0$,

$$\sum_{x\in\mathcal{S}} \pi'(x) = \frac{\sum_{x\in\mathcal{S}} p_x^{-1}\pi(x)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} = 1,$$

and for any $z \in \mathcal{S}$,

$$\begin{aligned}
(\pi' Q)(z) &= \sum_{x\in\mathcal{S}} \pi'(x) Q(x,z) \\
&= \frac{\sum_{x\in\mathcal{S}:x\neq z} p_x^{-1}\pi(x) p_x P(x,z) + p_z^{-1}\pi(z)(1-p_z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} \\
&= \frac{\sum_{x\in\mathcal{S}:x\neq z} \pi(x) P(x,z) - \pi(z) + p_z^{-1}\pi(z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} \\
&= \frac{(\pi P)(z) - \pi(z) + p_z^{-1}\pi(z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} \\
&= \frac{p_z^{-1}\pi(z)}{\sum_{y\in\mathcal{S}} p_y^{-1}\pi(y)} = \pi'(z).
\end{aligned}$$

Hence π' is the stationary distribution of the Markov chain with respect to the transition function Q.

17. Solution. Note that this chain is irreducible and positive recurrent and the stationary distribution is given by Q7(a):

$$\pi(n) = \frac{\binom{d}{n}}{2^d}, \quad 0 \le n \le d.$$

Hence the mean return time to state 0 is

$$m_0 = \frac{1}{\pi(0)} = 2^d$$

by Theorem 5 in page 64.

18. Solution. (a) Let $A = \{1, 2, \dots, c\}$ and $B = \{c+1, c+2, \dots, c+d\}$.
For $x, y \in A$, $\rho_{xy} \ge P(x, c+1)P(c+1, y) = (1/d)(1/c) > 0$.
For $x, y \in B$, $\rho_{xy} \ge P(x, 1)P(1, y) = (1/c)(1/d) > 0$.
For $x \in A$, $y \in B$, $\rho_{xy} \ge P(x, y) = 1/d > 0$ and $\rho_{yx} \ge P(y, x) > 0$.
Hence the chain is irreducible.

(b) Since the chain is irreducible and finite, it has a unique stationary distribution π.

For $y \in A$, we have

$$\pi(y) = (\pi P)(y) = \sum_{x \in B} \pi(x)P(x, y) = \frac{1}{c}\sum_{x \in B} \pi(x),$$

which implies

$$\sum_{y \in A} \pi(y) = \sum_{y \in A} \frac{1}{c} \sum_{x \in B} \pi(x) = \sum_{x \in B} \pi(x).$$

Note that $\sum_{x \in A \cup B} \pi(x) = 1$. Hence $\sum_{y \in A} \pi(y) = \sum_{x \in B} \pi(x) = 1/2$. Thus for any $y \in A$, $\pi(y) = \frac{1}{2c}$.

For $z \in B$, we have

$$\pi(z) = (\pi P)(z) = \sum_{x \in A} \pi(x)P(x, z) = \frac{1}{d}\sum_{x \in A} \pi(x) = \frac{1}{2d}.$$

Therefore, the stationary distribution is

$$\pi(x) = \begin{cases} \frac{1}{2c}, & x \in A, \\ \frac{1}{2d}, & x \in B. \end{cases}$$